The Lie algebroid Poisson sigma model

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ABSTRACT: The Poisson-Weil sigma model, worked out by us in [25, 26], stems from gauging a Hamiltonian Lie group symmetry of the target space of the Poisson sigma model. Upon gauge fixing of the BV master action, it yields interesting topological field theories such as the 2-dimensional Donaldson-Witten topological gauge theory and the gauged A topological sigma model. In this paper, generalizing the above construction, we construct the Lie algebroid Poisson sigma model. This is yielded by gauging a Hamiltonian Lie groupoid symmetry of the Poisson sigma model target space. We use the BV quantization approach in the AKSZ geometrical version to ensure consistent quantization and target space covariance. The model has an extremely rich geometry and an intricate BV cohomology, which are studied in detail.

Contents

1. Introduction

In geometry and in physics, symmetry is normally described in terms of groups and group actions. However, there are more general forms of symmetry, which do not let themselves be dealt with in that way, but which, nevertheless, are clearly recognizable as such. The mathematical structure that underlies them is that of groupoid and groupoid action.

The algebraic notion of groupoid was introduced by W. Brandt in 1926 as a generalization of that of group [1]. Since then, groupoids have found a wide range of mathematical applications. Topological and Lie groupoids, groupoids equipped with a compatible topological and differential structure, were used systematically by Ehresmann in his work in algebraic and differential geometry [2]. Groupoids were also employed in algebraic geometry by Grothendieck The notion of principal bundles with structure groupoid was worked out by Connes in the study of the holonomy groupoid of a foliation in [5]. Coste, Dazord, Weinstein, Karasev and Zakrzewski used symplectic groupoids, Lie groupoids equipped with a compatible symplectic structure, in the study of non commutative deformations of the algebra of smooth functions on a manifold [6 – 9]. In [10], Weinstein introduced Poisson

groupoids as generalizations of both Poisson Lie groups and symplectic groupoids. See [11] for a review of applications of groupoids in mathematics.

Lie algebroids were first studied by Pradines in the early sixties in relation with Lie groupoids. [12]. Since then, they have proved to provide a very general and flexible framework for studying a wide range of geometrical structures.

Lie algebroids are a vector bundle generalization of Lie algebras. A Lie algebra is just a Lie algebroid over a point. To any Lie groupoid there is associated a Lie algebroid much in the same way as to a Lie group there is associated a Lie algebra. Therefore, Lie algebroid theory parallels Lie groupoid theory as the infinitesimal version of the latter. However, the scope of Lie algebroids is broader, since, unlike what happens for ordinary Lie algebras, not every Lie algebroid is integrable to a Lie groupoid. The conditions for integrability were found in [13].

In recent years, Lie groupoids and Lie algebroids have attracted much interest also in theoretical physics because of their potential of describing the generalized forms of symmetry arising in the so called non linear gauge theories [14]. In ordinary linear gauge theories, the symmetries are local, the symmetry algebra closes off-shell and the symmetry algebra structure constants are field independent. By contrast, in non linear gauge theories, the symmetries are still local, but the symmetry algebra closes only on-shell and the symmetry algebra structure constants are field dependent (and, so, actually structure functions). While the symmetries of linear gauge theories are amenable by standard Lie theoretic techniques in an essentially finite dimensional setting, those of non linear gauge theories are not manifestly so. This renders the geometrical properties of non linear gauge theories rather mysterious and poses serious problems for their consistent quantization.

Lie groupoids and Lie algebroids constitute a promising framework for studying the symmetries of non linear gauge theories [15, 16]. They can accommodate the local symmetries of these theories while still allowing for an essentially finite dimensional treatment. Moreover, they reproduce the standard Lie theoretic framework in the linear case.

This approach, though completely general, has been adopted mostly in the study of the Poisson sigma model [17, 18], the prototype non linear gauge theory, and related models. In [19 – 21], the field equations of these field theories are interpreted as morphisms from the space-time tangent Lie algebroid to a certain target Lie algebroid and their on-shell symmetries as homotopies of such morphisms. In [22, 23], a Poisson sigma model with an integrable Poisson target manifold is considered, the field equations are interpreted as morphisms from the world sheet fundamental groupoid to the target manifold integrating groupoid and the symmetries are described in terms of an infinite dimensional groupoid of maps from the world sheet to this groupoid. In [24], these constructions have been interpreted in terms of the geometry of appropriate principal groupoid bundles. The present work is one more step in the same direction, but from a different perspective.

The Poisson-Weil sigma model, introduced by us in ref. [25] and further refined in [26] (see also [27]), is a Poisson sigma model in which a Hamiltonian Lie group symmetry of the target space is gauged. In [26], it is shown that, upon carrying out an appropriate gauge fixing, the Poisson-Weil model yields the 2-dimensional version of Donaldson-Witten topological gauge theory, describing the moduli space of flat connections on a closed sur-

face [28, 29], in the pure gauge case where the target space is a point, and the gauged A topological sigma model describing the moduli space of solutions of the so called vortex equations worked out by Baptista [30 – 32], in the case where the target space is a manifold with a Kaehler structure preserved by the symmetry action. In this paper, developing on the results of [25, 26], we construct the Lie algebroid Poisson sigma model. In simple terms, this is a Poisson sigma model in which a Hamiltonian Lie groupoid symmetry of the target space is gauged. In more precise terms, we consider the Poisson sigma model on a Poisson manifold X fibered over another manifold M and gauge the symmetry associated with a Hamiltonian infinitesimal action of a regular Lie algebroid L over M on X. Though we have in mind especially the case where L is the Lie algebroid AG of a Lie groupoid G, the model is consistently defined also when L is not integrable. The whole construction is conceived in such a way to reproduce the Poisson-Weil sigma model of refs. [25, 26] in the particular case where M is a point and L is an ordinary Lie algebq The model has a rich geometry. The kernel $\ker \rho$ of the anchor ρ of L plays a central role in the whole construction. The target space geometry involves a generalized moment map, which is defined only on sections of $\ker \rho$. When the Lie algebroid L integrates to a Lie groupoid G, the model encodes a generalized fiberwise Hamiltonian reduction of the target Poisson manifold X [33, 34].

We use the Batalin-Vilkoviski (BV) quantization approach [35, 36] in the geometrical version of Alexandrov, Kontsevich, Schwartz and Zaboronsky (AKSZ) [37], as developed by Cattaneo and Felder in [38, 39], to ensure consistent quantization and target space covariance. The BV cohomology is turns out to be quite intricate, being the total cohomology of a double complex interpolating the Lie algebroid cohomology complex of L and the tangential Poisson cohomology complex of X.

The hope of our endeavor is to find sensible gauge fixing prescriptions of the Lie algebroid Poisson sigma model yielding interesting topological field theories generalizing the ones mentioned above (see [23] for an attempt in this direction). We leave this for the future. On a broader perspective, our work is to be considered a theoretic experiment aimed to exploring possible new lines of development of quantum field theory.

The paper is organized as follows. In section 2, we review briefly the basic notions of the theory of Lie algebroids and Lie algebroid infinitesimal actions on fibered Poisson manifolds. In section 3, we illustrate the Lie algebroid Poisson sigma model and its main properties. In section 4, we review briefly various cohomologies associated to the target space geometry of the model. In section 5, relying on the results of section 4, we study in detail the BV cohomology. In section 6, after reviewing briefly the basic notions of the theory of Lie groupoids, we describe the generalized fiberwise Hamiltonian reduction encoded in the target space geometry of the model in the integrable case. In section 7, we illustrate a few examples. Section 8 contains some concluding remarks. Finally, in app. A–C, we conveniently collect some of the technical details of the analysis expounded in the main body of the paper.

Acknowledgments

2. Lie algebroids and their action on Poisson manifolds

In this section, we shall review briefly the basic notions of the theory of Lie algebroids and Lie algebroid infinitesimal actions [40–42], concentrating on actions on fibered Poisson manifolds because of their relevance on the following constructions [43]. We have expressed the relevant geometrical relations also in local coordinates to help the reader to check the calculation on his/her own.

A *Lie algebroid* is a vector bundle L over a manifold M equipped with a bundle map $\rho : L \to TM$, called the *anchor* and an $\mathbb{R}$-linear bracket $[\cdot,\cdot] : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ with the following properties.

1) $[\cdot,\cdot]$ is a Lie bracket so that $\Gamma(L)$ is a Lie algebra:

$$[s,t] + [t,s] = 0, \tag{2.1}$$

$$[s,[t,u]] + [t,[u,s]] + [u,[s,t]] = 0, \tag{2.2}$$

for $s,t,u \in \Gamma(L)$.

2) ρ defines a Lie algebra morphism of $\Gamma(L)$ into $\Gamma(TM)$:

$$\rho([s,t]) = [\rho(s),\rho(t)]_{TM}, \tag{2.3}$$

for $s,t \in \Gamma(L)$, where $[\cdot,\cdot]_{TM}$ is the usual Lie bracket of vector fields of M.

3) The Leibniz rule holds:

$$[s,ft] = f[s,t] + (l_{\rho(s)}f)t, \tag{2.4}$$

for $f \in C^\infty(M)$ and $s,t \in \Gamma(L)$, where l_v denotes Lie derivation along a vector field $v \in \Gamma(TM)$.

The prototype Lie algebroid over M is the tangent bundle TM: the anchor is the identity id_{TM} and the bracket is the usual Lie bracket $[\cdot,\cdot]_{TM}$. Lie algebroids generalize Lie algebras: a Lie algebra can be viewed as a Lie algebroid over the singleton manifold $M = \mathrm{pt}$.

Let $\{e_i\}$ be a local frame of L. Then, one has

$$\rho(e_i) = \rho_i{}^r \partial_r, \tag{2.5}$$

$$[e_i,e_j] = f^k{}_{ij} e_k. \tag{2.6}$$

$\rho_i{}^r$, $f^k{}_{ij}$ are called the anchor and structure functions of L, respectively. From (2.1)–(2.4), they satisfy

$$f^i{}_{jk} + f^i{}_{kj} = 0, \tag{2.7}$$

$$f^i{}_{jm}f^m{}_{kl} + f^i{}_{km}f^m{}_{lj} + f^i{}_{lm}f^m{}_{jk} + \rho_j{}^r\partial_r f^i{}_{kl} + \rho_k{}^r\partial_r f^i{}_{lj} + \rho_l{}^r\partial_r f^i{}_{jk} = 0, \tag{2.8}$$

$$\rho_i{}^s\partial_s\rho_j{}^r - \rho_j{}^s\partial_s\rho_i{}^r - f^k{}_{ij}\rho_k{}^r = 0. \tag{2.9}$$

A Lie algebroid L over M is said *regular*, if the anchor ρ has locally constant rank. In such a case, $\ker \rho$ is a bundle of Lie algebras, but not a Lie algebra bundle in general. If L is regular, $\Gamma(\ker \rho)$ is a Lie ideal of $\Gamma(L)$. A Lie algebroid is said *transitive*, if ρ is surjective. A transitive Lie algebroid is obviously regular and its $\ker \rho$ is a Lie algebra bundle.

For a regular Lie algebroid, we can choose adapted frames $\{e_i\} = \{e_\alpha\} \cup \{e_\kappa\}$, where $\{e_\alpha\}$ is a frame of $\ker \rho$. Such frames will be tacitly assumed, unless otherwise stated. Clearly, one has

$$\rho_\alpha{}^r = 0 \tag{2.10}$$

identically. Further,

$$f^\kappa{}_{i\alpha} = 0, \tag{2.11}$$

since $\Gamma(\ker \rho)$ is a Lie ideal.

A *(base preserving) morphism* of two Lie algebroids L, L' over M is a vector bundle morphism $\varphi : L \to L'$ such that

$$\rho' \circ \varphi = \rho, \tag{2.12}$$

$$\varphi([s,t]) = [\varphi(s), \varphi(t)]', \tag{2.13}$$

with $s, t \in \Gamma(L)$.

If K, L are two Lie algebroids over M and K is a subbundle of L, then K is a *subalgebroid* of L, if the natural injection $\iota : K \to L$ is a Lie algebroid morphism. If L is regular, then $K = \ker \rho$ is a subalgebroid of L.

We recall that a *fibered manifold* is a manifold X together with a surjective submersion $J : X \to M$ onto another manifold M.

Let L be a Lie algebroid over M and let $J : X \to M$ be a fibered manifold. An *infinitesimal action* of L on X along J is an $\mathbb{R}$-linear map $u : \Gamma(L) \to \Gamma(TX)$ with the following properties [42].

1) u is $C^\infty(M)$-linear:
$$u(fs) = (f \circ J)u(s), \tag{2.14}$$

   for $f \in C^\infty(M)$ and $s \in \Gamma(L)$.

2) u is a Lie algebra morphism:

$$u([s,t]) = [u(s), u(t)]_{TX}, \tag{2.15}$$

   for $s, t \in \Gamma(L)$.

3) u is projectable:

$$TJ(u(s)) = \rho(s) \circ J, \tag{2.16}$$

   for $s \in \Gamma(L)$, where TJ is the tangent map of J.

The last conditions implies that the vector fields $u(s)$ with $s \in \Gamma(\ker \rho)$ are tangent to the fibers of J.

Upon picking a frame $\{e_i\}$ of L, one has

$$u(e_i) = u_i{}^a \partial_a, \tag{2.17}$$

where $u_i{}^a$ are the Lie algebroid action functions. From (2.14)–(2.16), they satisfy:

$$u_i{}^b \partial_b u_j{}^a - u_j{}^b \partial_b u_i{}^a - f^k{}_{ij} \circ J u_k{}^a = 0, \tag{2.18}$$

$$u_i{}^a \partial_a J^r = \rho_i{}^r \circ J. \tag{2.19}$$

To an infinitesimal action of L on X along J, there is canonically associated a Lie algebroid structure on the pull back vector bundle J^*L. Its definition invokes the isomorphism $C^\infty(X) \otimes_{C^\infty(M)} \Gamma(L) \simeq \Gamma(J^*L)$ given by $f \otimes s \to f(s \circ J)$ with $f \in C^\infty(X)$, $s \in \Gamma(L)$. The anchor and Lie bracket are then defined by

$$\rho_J(f \otimes s) = f u(s), \tag{2.20}$$

$$[f \otimes s, g \otimes t]_J = fg \otimes [s,t] + (f l_{u(s)} g) \otimes t - (g l_{u(t)} f) \otimes s, \tag{2.21}$$

for $f, g \in C^\infty(X)$, $s, t \in \Gamma(L)$. The resulting Lie algebroid is called the *action Lie algebroid* corresponding to the infinitesimal action and is usually denoted by $L \ltimes J$. The anchor and structure functions of $L \ltimes J$ are $u_i{}^a$ and $f^i{}_{jk} \circ J$, respectively, as is easy to see.

A *Poisson structure* on a manifold X is a 2-vector field $P \in \Gamma(\wedge^2 TX)$ satisfying the Poisson condition

$$[P,P]_{\wedge^* TX} = 0, \tag{2.22}$$

where $[\cdot,\cdot]_{\wedge^* TX}$ is the Schouten-Nijenhuis bracket. The Poisson structure P allows the definition of a Poisson bracket on X by setting $\{f,g\}_P = P(df, dg)$, where $f, g \in C^\infty(X)$. X is thus called a *Poisson manifold*.

In local coordinates, P is given by

$$P = \frac{1}{2} P^{ab} \partial_a \wedge \partial_b. \tag{2.23}$$

Then, the Poisson condition (2.22) reads

$$P^{ad} \partial_d P^{bc} + P^{bd} \partial_d P^{ca} + P^{cd} \partial_d P^{ab} = 0. \tag{2.24}$$

A *fibered Poisson manifold* is a fibered manifold $J : X \to M$ together with a Poisson structure P on X satisfying the condition

$$P(TJ)^* = 0, \tag{2.25}$$

where we view $P \in \Gamma(\mathrm{Hom}(T^*X, TX))$. Intuitively, this means that the 2-vector field P is everywhere tangent to the fibers of J. X can then be viewed as a family of Poisson manifolds smoothly parametrized by M.

In local coordinates, (2.25) reads simply

$$P^{ab} \partial_b J^r = 0. \tag{2.26}$$

Let L be a Lie algebroid over M and $J : X \to M, P$ be a fibered Poisson manifold and let L act infinitesimally on X along J. P is said *invariant* if

$$l_{u(s)} P = 0, \qquad (2.27)$$

for $s \in \Gamma(L)$.

In local coordinates, the invariance condition (2.27) reads

$$u_i{}^c \partial_c P^{ab} - \partial_c u_i{}^a P^{cb} - \partial_c u_i{}^b P^{ac} = 0. \qquad (2.28)$$

Let L be a regular Lie algebroid over M and $J : X \to M, P$ be a fibered Poisson manifold and let L act infinitesimally on X along J leaving P invariant. An *equivariant moment map* for the action is an $\mathbb{R}$-linear map $\mu : \Gamma(\ker \rho) \to C^\infty(X)$ with the following properties [33, 34, 44].

1) μ is $C^\infty(M)$-linear:
$$\mu(fs) = (f \circ J)\mu(s), \qquad (2.29)$$

for $f \in C^\infty(M)$ and $s \in \Gamma(\ker \rho)$.

2) μ is equivariant:
$$l_{u(s)} \mu(t) = \mu([s, t]), \qquad (2.30)$$

for $s \in \Gamma(L)$ and $t \in \Gamma(\ker \rho)$.

3) μ is a moment map for u:
$$u(s) = \#_P d_X \mu(s), \qquad (2.31)$$

for $s \in \Gamma(\ker \rho)$, where $\#_P : T^*X \to TX$ is the sharp map associated to P by viewing $P \in \Gamma(\mathrm{Hom}(T^*X, TX))$. These relations imply that

$$\{\mu(s), \mu(t)\}_P = \mu([s, t]), \qquad (2.32)$$

for $s, t \in \Gamma(\ker \rho)$.

Pick an adapted frame $\{e_i\}$ of L and set

$$\mu_\alpha = \mu(e_\alpha). \qquad (2.33)$$

Then,(2.30), (2.31) in local coordinates read

$$u_\alpha{}^a + P^{ab} \partial_b \mu_\alpha = 0, \qquad (2.34)$$
$$u_i{}^a \partial_a \mu_\alpha - f^\beta{}_{i\alpha} \circ J \mu_\beta = 0. \qquad (2.35)$$

The fact that $\mu(s)$ is defined for $s \in \Gamma(\ker \rho)$ rather than $s \in \Gamma(L)$ may seem puzzling at first glance. In regard to this, let us note that, if $\mu(s)$ were defined for $s \in \Gamma(L)$, the equivariance condition (2.30) would not be covariant. Let us note further that, when L is an ordinary Lie algebra $\mathfrak{g}$, then $\ker \rho = L$ and one has $\mu : \mathfrak{g} \to C^\infty(X)$ as usual.

In the geometrical framework illustrated above, a pivotal role is played by the action of a Lie algebroid L over M on a fibered Poisson manifold $J : X \to M, P$. In this respect, there are two important extremal cases deserving mention.

a) $M = X$ and $J : X \to X$ the identity map. In this case, the infinitesimal action of L on X reduces to the canonical infinitesimal action of L on M, for which $u = \rho$. Further, P is necessarily trivial and thus trivially invariant under L (see eqs. (2.25), (2.27)).

b) $M = \mathrm{pt}$ and $J : X \to \mathrm{pt}$ the constant map. In this case, the infinitesimal action of L on X yields an ordinary infinitesimal action of the Lie algebra associated to L on X, whose fundamental vector field is u. Further, the Poisson structure P is subject only to the invariance condition under the Lie algebra action but is otherwise arbitrary (see again eqs. (2.25), (2.27)).

The general case is in a sense intermediate and interpolates between the two above extremal cases.

3. The Lie algebroid Poisson sigma model

In this section, we shall construct a sigma model canonically associated to the following geometrical data (cf. section 2).

1. A regular Lie algebroid L over M.

2. A fibered Poisson manifold $J : X \to M, P$

3. An infinitesimal action of L on X along J leaving P invariant.

4. A equivariant moment map μ for the action.

We shall call it *Lie algebroid Poisson sigma model* for evident reasons. We shall use a BV formalism [35, 36] following the geometrical approach of AKSZ [37] and Cattaneo and Felder [38, 39].

The base space of the model is the parity shifted tangent bundle $T[1]\Sigma$ of a closed surface Σ, the world sheet. The target space of the model is a graded manifold, the parity shifted vector bundle over X

$$\mathfrak{X}_{L,J} = (J^*L)^*[0] \oplus (J^* \ker \rho)^*[-1]. \tag{3.1}$$

The fields of the model organize in a superfield $\boldsymbol{\Phi} \in C^\infty(T[1]\Sigma, T^*[1]\mathfrak{X}_{L,J})$, where $T^*[1]\mathfrak{X}_{L,J}$ is the parity shifted cotangent bundle of $\mathfrak{X}_{L,J}$. Locally in target space, $\boldsymbol{\Phi}$ is given as a sextuplet of superfields $(\boldsymbol{x}^a, \boldsymbol{b}_i, \boldsymbol{B}_\alpha, \boldsymbol{y}_a, \boldsymbol{c}^i, \boldsymbol{C}^\alpha)$ of degrees $(0, 0, -1, 1, 1, 2)$, respectively. The triples $(\boldsymbol{x}^a, \boldsymbol{b}_i, \boldsymbol{B}_\alpha)$, $(\boldsymbol{y}_a, \boldsymbol{c}^i, \boldsymbol{C}^\alpha)$ correspond to the base and fiber coordinates of $T^*[1]\mathfrak{X}_{L,J}$, respectively. In turn, $\boldsymbol{x}^a$, $(\boldsymbol{b}_i, \boldsymbol{B}_\alpha)$ correspond to the base and fiber coordinates of $\mathfrak{X}_{L,J}$. We note that $\boldsymbol{x} \in C^\infty(T[1]\Sigma, X)$ and that $\boldsymbol{b} \in \Gamma(\boldsymbol{x}^*((J^*L)^*[0]))$, $\boldsymbol{B} \in \Gamma(\boldsymbol{x}^*((J^* \ker \rho)^*[-1]))$, $\boldsymbol{c} \in \Gamma(\boldsymbol{x}^*(J^*L[1]))$, $\boldsymbol{C} \in \Gamma(\boldsymbol{x}^*(J^* \ker \rho[2]))$, while $\boldsymbol{y}$ does not have an analogous simple interpretation. See app. A for details on covariance for the manifold $T^*[1]\mathfrak{X}_{L,J}$.

The field space is equipped with a degree -1 symplectic form obtained by pulling back with the evaluation map of $C^\infty(T[1]\Sigma, T^*[1]\mathfrak{X}_{L,J})$ the canonical symplectic form of $T^*[1]\mathfrak{X}_{L,J}$ and then integrating over $T[1]\Sigma$:

$$\Omega_{L,J} = \int_{T[1]\Sigma} \varrho \Big[\delta \boldsymbol{x}^a \delta \boldsymbol{y}_a + \delta \boldsymbol{b}_i \delta \boldsymbol{c}^i + \delta \boldsymbol{B}_\alpha \delta \boldsymbol{C}^\alpha \Big], \tag{3.2}$$

where ϱ is the invariant supermeasure on $T[1]\Sigma$. From this, one obtains the BV antibracket $(\cdot,\cdot)_{L,X}$ in standard fashion:

$$(F,G)_{L,J} = \int_{T[1]\Sigma} \varrho \left[\frac{\delta_r F}{\delta \boldsymbol{x}^a} \frac{\delta_l G}{\delta \boldsymbol{y}_a} - \frac{\delta_r F}{\delta \boldsymbol{y}_a} \frac{\delta_l G}{\delta \boldsymbol{x}^a} + \frac{\delta_r F}{\delta \boldsymbol{b}_i} \frac{\delta_l G}{\delta \boldsymbol{c}^i} \right. \tag{3.3}$$
$$\left. - \frac{\delta_r F}{\delta \boldsymbol{c}^i} \frac{\delta_l G}{\delta \boldsymbol{b}_i} + \frac{\delta_r F}{\delta \boldsymbol{B}_\alpha} \frac{\delta_l G}{\delta \boldsymbol{C}^\alpha} - \frac{\delta_r F}{\delta \boldsymbol{C}^\alpha} \frac{\delta_l G}{\delta \boldsymbol{B}_\alpha} \right],$$

where $\delta_{l,r}/\delta\boldsymbol{\phi}$ denotes left/right functional derivation with respect to the superfield $\boldsymbol{\phi}$.

The action of the model is

$$S_{J,P} = \int_{T[1]\Sigma} \varrho \left[\boldsymbol{y}_a \big(d\boldsymbol{x}^a + u_i{}^a(\boldsymbol{x})\boldsymbol{c}^i \big) + \mu_\alpha(\boldsymbol{x})\boldsymbol{C}^\alpha + \frac{1}{2}P^{ab}(\boldsymbol{x})\boldsymbol{y}_a\boldsymbol{y}_b \right. \tag{3.4}$$
$$\left. + \boldsymbol{b}_i\left(d\boldsymbol{c}^i - \frac{1}{2}f^i{}_{jk}(J(\boldsymbol{x}))\boldsymbol{c}^j\boldsymbol{c}^k + \delta^i{}_\alpha\boldsymbol{C}^\alpha \right) - \boldsymbol{B}_\alpha\big(d\boldsymbol{C}^\alpha - f^\alpha{}_{i\beta}(J(\boldsymbol{x}))\boldsymbol{c}^i\boldsymbol{C}^\beta \big) \right].$$

For the target space global definedness of the integrand, it is absolutely crucial that the Poisson structure P satisfies the tangentiality condition (2.26). This follows straightforwardly from eqs. (A.1) in app. A. Above, one views $u \in \Gamma(\text{Hom}(J^*L, TX))$ and $\mu \in \Gamma((J^*\ker\rho)^*)$, as allowed by (2.14), (2.29).

The properties of the target space geometry of the sigma model make $S_{J,P}$ satisfy the BV classical master equation [35, 36]

$$(S_{J,P}, S_{J,P})_{L,J} = 0. \tag{3.5}$$

The verification is a straightforward calculation exploiting certain combinations of the local coordinate relations (2.7), (2.8), (2.10), (2.11), (2.18), (2.19), (2.24), (2.26), (2.28), (2.34), (2.35). We observe that these relations are sufficient but not necessary conditions for the validity of (3.5). This fact is a recurrent feature of the AKSZ formulation of sigma models.

Associated with the master action $S_{J,P}$ is the BV field variation operator $\delta_{J,P} := (S_{J,P}, \cdot)_{L,J}$. The BV field variations are:

$$\delta_{J,P}\boldsymbol{x}^a = d\boldsymbol{x}^a + u_i{}^a(\boldsymbol{x})\boldsymbol{c}^i + P^{ab}(\boldsymbol{x})\boldsymbol{y}_b, \tag{3.6a}$$

$$\delta_{J,P}\boldsymbol{y}_a = d\boldsymbol{y}_a + \partial_a u_i{}^b(\boldsymbol{x})\boldsymbol{y}_b\boldsymbol{c}^i + \partial_a\mu_\alpha(\boldsymbol{x})\boldsymbol{C}^\alpha + \frac{1}{2}\partial_a P^{bc}(\boldsymbol{x})\boldsymbol{y}_b\boldsymbol{y}_c \tag{3.6b}$$
$$- \frac{1}{2}\partial_a J^r(\boldsymbol{x})\partial_r f^i{}_{jk}(J(\boldsymbol{x}))\boldsymbol{b}_i\boldsymbol{c}^j\boldsymbol{c}^k - \partial_a J^r(\boldsymbol{x})\partial_r f^\alpha{}_{i\beta}(J(\boldsymbol{x}))\boldsymbol{c}^i\boldsymbol{B}_\alpha\boldsymbol{C}^\beta,$$

$$\delta_{J,P}\boldsymbol{c}^i = d\boldsymbol{c}^i - \frac{1}{2}f^i{}_{jk}(J(\boldsymbol{x}))\boldsymbol{c}^j\boldsymbol{c}^k + \delta^i{}_\alpha\boldsymbol{C}^\alpha, \tag{3.6c}$$

$$\delta_{J,P}\boldsymbol{b}_i = d\boldsymbol{b}_i + f^j{}_{ki}(J(\boldsymbol{x}))\boldsymbol{b}_j\boldsymbol{c}^k + f^\alpha{}_{\beta i}(J(\boldsymbol{x}))\boldsymbol{B}_\alpha\boldsymbol{C}^\beta - u_i{}^a(\boldsymbol{x})\boldsymbol{y}_a, \tag{3.6d}$$

$$\delta_{J,P}\boldsymbol{C}^\alpha = d\boldsymbol{C}^\alpha - f^\alpha{}_{i\beta}(J(\boldsymbol{x}))\boldsymbol{c}^i\boldsymbol{C}^\beta, \tag{3.6e}$$

$$\delta_{J,P}\boldsymbol{B}_\alpha = d\boldsymbol{B}_\alpha + f^\beta{}_{i\alpha}(J(\boldsymbol{x}))\boldsymbol{c}^i\boldsymbol{B}_\beta - \boldsymbol{b}_\alpha - \mu_\alpha(\boldsymbol{x}). \tag{3.6f}$$

The master equation (3.5) implies that $S_{J,P}$ is invariant under $\delta_{J,P}$,

$$\delta_{J,P}S_{J,P} = 0 \tag{3.7}$$

and that $\delta_{J,P}$ is nilpotent

$$\delta_{J,P}{}^2 = 0. \tag{3.8}$$

It is interesting to examine what happens in the two extremal cases considered at the end of section 2.

a) In this case, $M = X$ and J is the identity map id_X. Since $P = 0$ identically, if we also set $\mu = 0$, we get a sigma model canonically associated to the Lie algebroid L, which we call *Lie algebroid sigma model*. It is simple to check that the basic relations obeyed by the anchor and the structure functions, eqs. (2.8), (2.9), are not only sufficient but also necessary for the BV classical master equation (3.5) to hold.

b) In this case, $M = \mathrm{pt}$ and J is the constant map. L is an ordinary Lie algebra acting infinitesimally on X and P is invariant under such an action. The resulting sigma model is nothing but the Poisson-Weil model of refs. [25, 26] for trivial twisting principal bundle. See section 8 for more on this point.

For general M and J, the Lie algebroid Poisson sigma model is consistently defined for $P = 0$ and $\mu = 0$ provided $u(s) = 0$ for $s \in \Gamma(\ker \rho)$. (In general, only the weaker condition $TJ(u(s)) = 0$ holds, see eq. (2.16).) In that case, it reduces into the Lie algebroid sigma model of the action Lie algebroid $L \ltimes J$ (cf. section 2).

Inspection of the action (3.4) reveals that the Lie algebroid sigma model is a Poisson sigma model on the graded manifold $\mathfrak{X}_{L,J}$ twisted by a moment map potential term. The target space 2-vector $\Pi \in \Gamma(\wedge^2 T\mathfrak{X}_{L,J})$ of this Poisson sigma model is given by the following expressions:

$$\Pi^{ab}(\Xi) = P^{ab}(\xi), \tag{3.9a}$$

$$\Pi^a{}_i(\Xi) = u_i{}^a(\xi), \tag{3.9b}$$

$$\Pi^a{}_\alpha(\Xi) = 0, \tag{3.9c}$$

$$\Pi_{ij}(\Xi) = -f^k{}_{ij}(J(\xi))\beta_k, \tag{3.9d}$$

$$\Pi_{i\alpha}(\Xi) = -f^\beta{}_{i\alpha}(J(\xi))\mathrm{B}_\beta, \tag{3.9e}$$

$$\Pi_{\alpha\beta}(\Xi) = 0, \tag{3.9f}$$

where ξ^a, $(\beta_i, \mathrm{B}_\alpha)$ are respectively the base and fiber coordinates of the bundle $\mathfrak{X}_{L,J} = (J^*L)^*[0] \oplus (J^* \ker \rho)^*[-1]$ and we have set $\Xi^A = (\xi^a, \beta_i, \mathrm{B}_\alpha)$ (cf. app. A). The relations (2.7), (2.8), (2.10), (2.11), (2.18), (2.19), (2.24), (2.26), (2.28), which ensure the fulfillment of the BV classical master equation (3.5), also ensure that the 2-vector Π satisfies the Poisson condition.

4. Action Lie algebroid and Poisson cohomology

The Lie algebroid Poisson sigma model introduced in section 3 is characterized by the associated BV cohomology [40, 41]. This in turn is intimately related with various cohomologies associated with the target space geometry. In this section, we briefly review them.

A Lie algebroid L over M is endowed with a natural cohomology, the *Lie algebroid cohomology*. This is the cohomology of the complex $(A^*(L), d_L)$, where $A^p(L) = \Gamma(\wedge^p L^*)$ consists of $C^\infty(M)$-multilinear antisymmetric maps $\omega : \Gamma(L)^p \to C^\infty(M)$ and the nilpotent differential $d_L : A^p(L) \to A^{p+1}(L)$ is given by the well-known Chevalley-Eilenberg formula:

$$(d_L\omega)(s_1,\dots,s_{p+1}) = \sum_i (-1)^{i+1} l_{\rho(s_i)}(\omega(s_1,\dots,\hat{s}_i,\dots,s_{p+1})) \qquad (4.1)$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([s_i,s_j],s_1,\dots,\hat{s}_i,\dots,\hat{s}_j,\dots s_{p+1}),$$

with $s_1,\dots,s_{p+1} \in \Gamma(L)$.

The Lie algebroid cohomology complex can be described alternatively in supergeometric terms as follows. There is an isomorphism $\Gamma(\wedge^* L^*) \simeq C^\infty(L[1])$ defined by $\omega \to \frac{1}{p!}\omega(\xi,\dots,\xi)$ with $\omega \in \Gamma(\wedge^p L^*)$, where $\xi = \xi^i \otimes e_i$, e_i and ξ^i being the elements of a local frame of L and the corresponding degree 1 fiber coordinates of $L[1]$, respectively. Under the isomorphism, the differential d_L turns into the homological vector field over $L[1]$ given by

$$d_L = \xi^i l_{\rho(e_i)} - \frac{1}{2} f^k{}_{ij} \xi^i \xi^j \partial_{\xi k}, \qquad (4.2)$$

where $\partial_{\xi i} = \partial/\partial \xi^i$. The supergeometric formulation is more convenient in general.

A *representation* of a Lie algebroid L over M is a vector bundle E over M together with an assignment to each $s \in \Gamma(L)$ of an $\mathbb{R}$-linear map $D_s : \Gamma(E) \to \Gamma(E)$ with the following properties:

$$D_{fs}\sigma = f D_s\sigma, \qquad (4.3a)$$
$$D_s(f\sigma) = f D_s\sigma + (l_{\rho(s)}f)\sigma, \qquad (4.3b)$$
$$[D_s, D_t]\sigma = D_{[s,t]}\sigma, \qquad (4.3c)$$

where $s,t \in \Gamma(L)$, $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$. The trivial representations is defined by $E = M \times \mathbb{R}$ and $D_s h = l_{\rho(s)}h$, with $s \in \Gamma(L)$ and $h \in C^\infty(M)$. If L is regular (cf. section 2), the adjoint representation is defined by $E = \ker \rho$ and $D_s u = [s,u]$, with $s \in \Gamma(L)$ and $u \in \Gamma(\ker \rho)$. If the base M is a point, then L is a Lie algebra and E is a vector space, and a representation of L is just an ordinary Lie algebra linear representation.

One can define the Lie algebroid cohomology of L with values in a given representation D of L. This is the cohomology of the complex $(A^*(L,D), d_{L,D})$, where $A^p(L,D) = \Gamma(\wedge^p L^* \otimes E)$ consists of $C^\infty(M)$-multilinear antisymmetric maps $\omega : \Gamma(L)^p \to \Gamma(E)$ and the nilpotent differential $d_{L,D} : A^p(L,D) \to A^{p+1}(L,D)$ is given by the Chevalley-Eilenberg formula (4.1) with $l_{\rho(s_i)}$ replaced by D_{s_i}. There is also a supergeometric formulation exploiting the isomorphism $\Gamma(\wedge^* L^* \otimes E) \simeq \Gamma(\pi_{L[1]}{}^* E)$, where $\pi_{L[1]} : L[1] \to M$ is the bundle projection, in which $d_{L,D}$ turns into a homological vector field over $L[1]$ given by (4.2) with $l_{\rho(e_i)}$ replaced by D_{e_i}. When $E = M \times \mathbb{R}$ and $D_s = l_{\rho(s)}$, one recovers the usual Lie algebroid cohomology.

Let a Lie algebroid L over M act infinitesimally on a fibered manifold $J : X \to M$ (cf. section 2). To the action, one can associate the *action Lie algebroid cohomology*, the

Lie algebroid cohomology of the action Lie algebroid $L \ltimes J$ (cf. section 2). The associated cochain complex can be described as follows. The cochain space $A^p(L \ltimes J)$ consists of the antisymmetric maps $\omega : \Gamma(L)^p \to C^\infty(X)$ which are $C^\infty(M)$-multilinear meaning that

$$\omega(s_1, \ldots, fs_m, \ldots s_p) = (f \circ J)\omega(s_1, \ldots, s_m, \ldots s_p), \tag{4.4}$$

with $s_1, \ldots, s_p \in \Gamma(L)$ and $f \in C^\infty(M)$. The differential $d_{L \ltimes J}$ is then given by the Chevalley-Eilenberg formula (4.1) with $\rho(s_i)$ substituted by $u(s_i)$. In the supergeometric formulation, $d_{L \ltimes J}$ is given by (4.2) with $l_{\rho(e_i)}$ replaced by $l_{u(e_i)}$ and $f^k{}_{ij}$ by $f^k{}_{ij} \circ J$.

For analogous reasons, a representation D of $L \ltimes J$ on a vector bundle E over X can be described as an assignment to each $s \in \Gamma(L)$ of an $\mathbb{R}$-linear map $D_s : \Gamma(E) \to \Gamma(E)$ satisfying

$$D_{fs}\sigma = (f \circ J)D_s\sigma, \tag{4.5}$$

for $s \in \Gamma(L)$, $f \in C^\infty(M)$ and $\sigma \in \Gamma(E)$, in substitution of (4.3a), together with (4.3b), (4.3c) with $s, t \in \Gamma(L)$, $f \in C^\infty(X)$ and $\sigma \in \Gamma(E)$ and $\rho(s)$ replaced by $u(s)$. The action Lie algebroid cohomology with values in E can then be described as follows. The cochain space $A^p(L \ltimes J, D)$ consists of the antisymmetric maps $\omega : \Gamma(L)^p \to \Gamma(E)$ which are $C^\infty(M)$-multilinear in the sense (4.4). The differential $d_{L \ltimes J, D}$ is then given again by (4.1) with $l_{\rho(s_i)}$ replaced by D_{s_i}. Similarly, in the supergeometric formulation, $d_{L \ltimes J, D}$ is given again by (4.2) with $l_{\rho(e_i)}$ replaced by D_{e_i} and $f^k{}_{ij}$ by $f^k{}_{ij} \circ J$.

To an action of L on X along J, there is canonically associated a representation of the action Lie algebroid $L \ltimes J$ defined as follows. Let $T^J X = \ker TJ$. Since J is a submersion, $T^J X$ is a vector subbundle of TX. Then, letting $E = T^J X$, $D_s v = l_{u(s)}v$, with $s \in \Gamma(L)$ and $v \in \Gamma(T^J X)$, defines a representation D_1 of $L \ltimes J$. The restriction to $T^J X$ is required by the fulfillment of (4.5). In the same way, one can construct more general representations D_q of $L \ltimes J$ with $E = \wedge^q T^J X$. These are the only representations, which we shall consider in the following.

Let P be a Poisson structure on X (cf. section 2). As is well-known, P is characterized its *Poisson cohomology*. This is the cohomology of the complex $(V^*(X), d_P)$, where $V^q(X) = \Gamma(\wedge^q TX)$ is the space of q-vector fields and the Lichnerowicz differential $d_P : V^q(X) \to V^{q+1}(X)$ is defined by

$$d_P U = -[P, U]_{\wedge^* TX}, \tag{4.6}$$

$[\cdot, \cdot]_{\wedge^* TX}$ being the Schouten-Nijenhuis bracket. The cotangent bundle $T^* X$ of X has a canonical Lie algebroid structure associated to the Poisson structure P [43]. The Poisson cohomology of X equals the Lie algebroid cohomology of $T^* X$.

Let $J : X \to M, P$ be a fibered Poisson manifold (cf. section 2). Then, by (2.25), $P \in \Gamma(\wedge^2 T^J X)$. Define $V_J^q(X) = \Gamma(\wedge^q T^J X)$. Then, $(V_J^*(X), d_P)$ is a subcomplex of the complex $(V^*(X), d_P)$ and, thus, itself a complex. Its cohomology is the *tangential Poisson cohomology*. Here, the term "tangential" refers to the foliation of X induced by J.

Suppose that a Lie algebroid L over M acts infinitesimally on a fibered Poisson manifold $J : X \to M, P$ leaving P invariant (cf. section 2). Define $A_J^{p,q}(L) = \Gamma(\wedge^p(J^*L)^* \otimes \wedge^q T^J X)$. For fixed q, $A_J^{p,q}(L) = A^p(L \ltimes J, D_q)$. Setting $d_{J,L} = d_{L \ltimes J, D_q}$, one has that

$(A_{J}{}^{*,q}(L), d_{J,L})$ is a cochain complex. For fixed p, if $\omega \in A_{J}{}^{p,q}(P)$, then $d_P\omega \in A_{J}{}^{p,q+1}(P)$, as is easy to verify using again (2.25). Thus, $(A_{J}{}^{p,*}(L), d_P)$ is cochain complex. It can be verified that $d_P d_{J,L} + d_{J,L} d_P = 0$. It follows that $(A_{J}{}^{*,*}(L), d_{J,L}, d_P)$ is a double cochain complex. We call the associated cohomology the *action Lie algebroid Poisson cohomology* of $L, J : X \rightarrow M, P$. The cohomologies of $(A_{J}{}^{*,0}(L), d_{J,L})$, $(A_{J}{}^{0,*}(L), d_P)$ are the action Lie algebroid cohomology of L and the tangential Poisson cohomology of P, respectively. The total action Lie algebroid Poisson cohomology is the cohomology of the complex $(A_{J}{}^{*}(L), d_{J,L,P})$, where $A_{J}{}^{*}(L)$ is the complex $A_{J}{}^{*,*}(L)$, graded according to total degree, and $d_{J,L,P} = d_{J,L} + d_P$ is the total differential. See app. C for a supergeometric description of the double complex $(A_{J}{}^{*,*}(L), d_{J,L}, d_P)$.

5. BV cohomology of the Lie algebroid Poisson sigma model

The BV cohomology of the Lie algebroid Poisson sigma model is the cohomology of the nilpotent BV field variation operator $\delta_{J,P}$ (cf. eqs. (3.6)). Since our sigma model is essentially a Poisson sigma model on the graded manifold $\mathfrak{X}_{L,J}$ (cf. eq. (3.1)), one expects the BV cohomology to be related to the Poisson cohomology of the target space Poisson structure Π (cf. eq. (3.9)). One expects also there to be corrections due to the twisting by the moment map potential term. However, this point of view is not going to yield much in the way of detailed cohomological information. Therefore, we shall not pursue it any longer.

To bring to focus the relation of the BV cohomology with the target space geometry of the sigma model, it is convenient to consider, instead of the BV variation operator $\delta_{J,P}$, the mod $\boldsymbol{d}$ BV variation operator

$$\bar{\delta}_{J,P} = \delta_{J,P} - \boldsymbol{d}. \tag{5.1}$$

As $\delta_{J,P}$, $\bar{\delta}_{J,P}$ is nilpotent

$$\bar{\delta}_{J,P}{}^2 = 0. \tag{5.2}$$

The cohomology of $\bar{\delta}_{J,P}$ is the mod $\boldsymbol{d}$ BV cohomology and is the object of our study.

Because of the presence of the 0- and -1-degree superfields $\boldsymbol{b}_i$ and $\boldsymbol{B}_{\alpha}$, at each degree the most general superfield involves an infinite number of target space background fields. This renders the study of this cohomology problematic and not particularly illuminating. Fortunately, there is a subset $\mathfrak{X}^*$ of superfields that is interesting, on one hand, and is sufficiently restricted to allow for a simple study of the cohomology, on the other. $\mathfrak{X}^*$ consists of the superfields of the form

$$\boldsymbol{\Phi} = \sum_{p,h,q} \frac{1}{p!h!q!} \Phi_{(p,h,q)i_1\ldots i_p \alpha_1\ldots \alpha_h}{}^{a_1\ldots a_q}(\boldsymbol{x})\boldsymbol{c}^{i_1}\ldots \boldsymbol{c}^{i_p}\boldsymbol{C}^{\alpha_1}\ldots \boldsymbol{C}^{\alpha_h}\boldsymbol{y}_{a_1}\ldots \boldsymbol{y}_{a_q}, \tag{5.3}$$

where $\Phi_{(p,h,q)} \in \Gamma(\wedge^p(J^*L)^* \otimes \vee^h(J^* \ker \rho)^* \otimes \wedge^q T^J X)$. Restricting to $T^J X$ (cf. section 4) amounts to the condition

$$\Phi_{(p,h,q)i_1\ldots i_p \alpha_1\ldots \alpha_h}{}^{a_1\ldots a_{q-1}b}\partial_b J^r = 0. \tag{5.4}$$

(5.4) is required by the target space global definedness of the right hand side of (5.3), as follows easily from eqs. (A.1). It also implies that $\mathcal{X}^*$ is closed under the action of $\bar{\delta}_{J,P}$, as is apparent from eqs. (3.6). Thus, $\mathcal{X}^*$ is a subcomplex of the mod $\boldsymbol{d}$ BV cohomology superfield complex.

Using (3.6), one obtains straightforwardly the conditions on the $\Phi_{(p,h,q)}$ entailed by the mod $\boldsymbol{d}$ BV cocycle condition $\bar{\delta}_{J,P}\boldsymbol{\Phi} = 0$. The conditions are most naturally expressed by viewing the $\Phi_{(p,h,q)}$ as maps $\Phi_{(p,h,q)} : \Gamma(L)^p \times \Gamma(\ker\rho)^h \to \Gamma_J(\wedge^q T^J X)$ antisymmetric in the first p arguments and symmetric in the last h arguments and $C^\infty(M)$-linear in the same sense as (4.4). However, the resulting expressions are not very illuminating in the general case and, so, we shall not write them down explicitly. Rather, we shall consider the first few low degree cases, because of their special interest.

Degree 0. If $\boldsymbol{\Phi} \in \mathcal{X}^0$, then it is of the form

$$\boldsymbol{\Phi} = \phi(\boldsymbol{x}), \tag{5.5}$$

where $\phi \in C^\infty(X)$. Imposing $\bar{\delta}_{J,P}\boldsymbol{\Phi} = 0$ leads to the equations

$$\#_P d\phi = 0, \tag{5.6a}$$
$$l_{u(s)}\phi = 0, \tag{5.6b}$$

with $s \in \Gamma(L)$.

From a cohomological point of view, (5.6) states that ϕ is a 0-cocycle of the total action Lie algebroid Poisson cohomology complex (cf. section 4). In more conventional terms, ϕ is a Casimir function of the Poisson structure P invariant under the action of L.

In the extremal case a of section 2, (5.6a) is trivially satisfied as $P = 0$ and (5.6b) reduces into $l_{\rho(s)}\phi = 0$. In the extremal case b, (5.6) states that ϕ is a Casimir function invariant under the action of the Lie algebra associated to L.

Degree 1. If $\boldsymbol{\Phi} \in \mathcal{X}^1$, then it is of the form

$$\boldsymbol{\Phi} = w^a(\boldsymbol{x})\boldsymbol{y}_a + \sigma_i(\boldsymbol{x})\boldsymbol{c}^i, \tag{5.7}$$

where $w \in \Gamma(T^J X)$, $\sigma \in \Gamma((J^*L)^*)$. Imposing $\bar{\delta}_{J,P}\boldsymbol{\Phi} = 0$ leads to a set of equations, which can be cast as

$$-[P, w]_{\wedge^* TX} = 0, \tag{5.8a}$$
$$l_{u(s)}w - \#_P d\sigma(s) = 0, \tag{5.8b}$$
$$l_{u(s)}\sigma(t) - l_{u(t)}\sigma(s) - \sigma([s,t]) = 0, \tag{5.8c}$$
$$l_w\mu(z) + \sigma(z) = 0, \tag{5.8d}$$

with $s, t \in \Gamma(L)$ and $z \in \Gamma(\ker\rho)$.

In cohomological terms, (5.8a)–(5.8c) state that (w, σ) is a 1-cocycle of the total action Lie algebroid Poisson cohomology complex. In particular, by (5.8a), w is a 1-cocycle of the tangential Poisson cohomology complex of P and, by (5.8c), σ is a 1-cocycle of the

action Lie algebroid cohomology complex (cf. section 4). (5.8d) is a "boundary condition" determining $\sigma(z)$ for $z \in \Gamma(\ker \rho)$. More conventionally, since $[w, P]_{\wedge^* TX} = l_w P$, (5.8a) states that w is a Poisson vector field of the Poisson structure P, i.e. a vector field whose flow leaves P invariant. When $\sigma = 0$, the flow leaves also invariant the moment map μ and the Lie algebroid action vector fields $u(s)$ for $s \in \Gamma(L)$.

In the extremal case a of section 2, one has not only that $P = 0$ but also that $w = 0$, by (5.4). Thus, (5.8a), (5.8b) are trivially satisfied. Further, by (5.8c), being $u(s) = \rho(s)$, σ is a 1-cocycle of the Lie algebroid cohomology complex and, by (5.8d), the restriction of σ to $\Gamma(\ker \rho)$ is trivial. In the extremal case b, (5.8a) is the only non trivial condition. Indeed, as $\ker \rho = L$, $\sigma(s)$ is expressed in terms of w and $\mu(s)$ for all $s \in \Gamma(L)$ by (5.8d) and (5.8b), (5.8c) are automatically satisfied if (5.8a) is.

Degree 2. If $\mathbf{\Phi} \in \mathfrak{X}^2$, then it is of the form

$$\mathbf{\Phi} = \frac{1}{2} Q^{ab}(\boldsymbol{x}) \boldsymbol{y}_a \boldsymbol{y}_b - v_i{}^a(\boldsymbol{x}) \boldsymbol{c}^i \boldsymbol{y}_a + \frac{1}{2} \tau_{ij}(\boldsymbol{x}) \boldsymbol{c}^i \boldsymbol{c}^j + \nu_\alpha(\boldsymbol{x}) \boldsymbol{C}^\alpha, \qquad (5.9)$$

in which $Q \in \Gamma(\wedge^2 T^J X)$, $v \in \Gamma((J^* L)^* \otimes T^J X)$, $\tau \in \Gamma(\wedge^2 (J^* L)^*)$ and $\nu \in \Gamma((J^* \ker \rho)^*)$. Imposing $\bar{\delta}_{J,P} \mathbf{\Phi} = 0$ leads to the equations

$$-[P, Q]_{\wedge^* TX} = 0, \qquad (5.10\text{a})$$

$$l_{u(s)} Q - [P, v(s)]_{\wedge^* TX} = 0, \qquad (5.10\text{b})$$

$$-l_{u(s)} v(t) + l_{u(t)} v(s) + v([s,t]) + \#_P d\tau(s,t) = 0, \qquad (5.10\text{c})$$

$$l_{u(r)} \tau(s,t) - l_{u(s)} \tau(r,t) + l_{u(t)} \tau(r,s) - \tau([r,s],t) + \tau([r,t],s) - \tau([s,t],r) = 0, \qquad (5.10\text{d})$$

$$\#_Q d\mu(z) + \#_P d\nu(z) - v(z) = 0, \qquad (5.10\text{e})$$

$$l_{u(s)} \nu(z) + l_{v(s)} \mu(z) - \nu([s,z]) - \tau(s,z) = 0, \qquad (5.10\text{f})$$

where $r, s, t \in \Gamma(L)$ and $z \in \Gamma(\ker \rho)$.

From a cohomological point of view, (5.10a)–(5.10d) state that $(Q, -v, \tau)$ is a 2-cocycle of the total action Lie algebroid Poisson cohomology. In particular, by (5.10a), Q is a 2-cocycle of the tangential Poisson cohomology complex of P and, by (5.10d), τ is a 2-cocycle of the action Lie algebroid cohomology complex. (5.10e), (5.10f) are boundary conditions determining $v(z)$ and $\tau(s,z)$ for $s \in \Gamma(L)$ and $z \in \Gamma(\ker \rho)$. In more conventional terms, setting $P' = P + Q$, $u'(s) = u(s) + v(s)$ and $\mu'(z) = \mu(z) + \nu(z)$, (5.10a), (5.10b), (5.10e) state that P' is a Poisson structure invariant under the flow of the vector fields $u'(s)$ and that μ' is a (non equivariant) moment map for u' to linear order in Q, v and ν (cf. eqs. (2.22)),(2.27), (2.31)). When $\tau = 0$, (5.10c), (5.10f) state further that u' defines a new action of L on X along J and that the moment map μ' is equivariant under the new action again to linear order in Q, v and ν (cf. eqs. (2.15), (2.30)).

In the extremal case a of section 2, one has not only that $P = 0$ but also that $Q = 0$, $v = 0$, by (5.4). Thus, (5.10a), (5.10b), (5.10c), (5.10e) are trivially satisfied. Further, by (5.10d), being $u(s) = \rho(s)$, τ is a 2-cocycle of the Lie algebroid cohomology complex and, by (5.10f), the restriction of τ to $\Gamma(\ker \rho)$ is trivial. In the extremal case b, (5.10a) is the only non trivial condition. Indeed, as $\ker \rho = L$, $v(s)$, $\tau(s,t)$ are expressed in terms

of Q, $\nu(s)$ and P, $\mu(s)$ for all $s, t \in \Gamma(L)$ by (5.10e), (5.10f) and (5.10b)–(5.10d) are automatically satisfied if (5.8a) is.

The mod $\boldsymbol{d}$ BV cohomology in higher degree is expected to exhibit a similar structure. As usual, the degree 1 and 2 mod $\boldsymbol{d}$ cohomologies relate to the infinitesimal symmetries and infinitesimal deformations of the target space geometry, respectively. Strictly speaking this holds only when the action Lie algebroid 1- and 2-cocycles σ and τ above vanish. The interpretation of σ and τ in the general case is as yet unclear and calls for further investigation.

It is interesting to compare the mod $\boldsymbol{d}$ BV cohomology of the Lie algebroid Poisson sigma model in the case where M is a point and L is a Lie algebra with that of the Poisson-Weil sigma model studied in ref. [26]. To begin with, one must recall that, in the situation considered, the Lie algebroid Poisson sigma model reproduces the Poisson-Weil sigma model for a trivial twisting principal bundle (cf. section 3). In the Poisson-Weil model, in the general case, the superfield $\boldsymbol{c}^i$ is a generalized connection and must be absent in any expansion of the form (5.3) to have a superfield globally defined on the worldsheet. Further, the coefficients of the expansion must be covariant under the action of the symmetry Lie group to have a superfield invariantly defined in target space. For this reason, the analysis of ref. [26] was limited to the sector of the mod $\boldsymbol{d}$ BV cohomology complex formed by the superfields of the form (5.3) with no $\boldsymbol{c}^i$ factors and covariant coefficients.[1] Therefore, the comparison of Lie algebroid Poisson sigma model and Poisson-Weil sigma model mod $\boldsymbol{d}$ BV cohomologies can be carried out at best only upon restricting to a suitable sector of the former, that spanned by the superfields $\boldsymbol{\Phi}$ of the form (5.3) with no $\boldsymbol{c}^i$ occurrences. By (5.7), in degree 1, this amounts to imposing that $\sigma = 0$. Inspection of the 1-cocycle condition (5.8) shows that $\boldsymbol{\Phi}$ is a 1-cocycle of the equivariant Poisson cohomology (in the Cartan model), as found in [26]. Similarly, by (5.9), in degree 2, one must have $v = 0$ and $\tau = 0$ and the 2-cocycle condition (5.10) shows that $\boldsymbol{\Phi}$ is a 2-cocycle of the equivariant Poisson cohomology, again as found in [26].

6. Hamiltonian Lie groupoid actions and Poisson reduction

The notion of Lie groupoid is related to that of Lie algebroid in the same way as the notion of Lie group is related to that of Lie algebra [40, 41]. Unlike what happens for Lie algebras and groups, not all Lie algebroids integrate to a Lie groupoid. In this section, we review briefly the theory of Lie groupoids and their associated Lie algebroids and of Hamiltonian actions of Lie groupoids on fibered Poisson manifolds.

A *groupoid* consists of two sets G and M and five maps $\alpha : G \to M$, $\beta : G \to M$, $1 : M \to G$, $\iota : G \to G$, $\mu : G\,_\alpha\times_\beta G \to G$, where $G\,_\alpha\times_\beta G = \{(g, h) \in G \times G | \alpha(g) = \beta(h)\}$ with the following properties.

1) For $m \in M$, $\alpha(1_m) = \beta(1_m) = m$.

2) For $g \in G$, $\alpha(g^{-1}) = \beta(g)$ and $\beta(g^{-1}) = \alpha(g)$.

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[1]However, the analysis could have been generalized by introducing a fixed background generalized connection $\boldsymbol{A}^i$ and replacing $\boldsymbol{c}^i$ by $\boldsymbol{c}^i - \boldsymbol{A}^i$.

3) For $(g,h) \in G \,_\alpha\times_\beta G$, $\alpha(gh) = \alpha(h)$ and $\beta(gh) = \beta(g)$.

4) For $g \in G$, $g1_{\alpha(g)} = 1_{\beta(g)}g = g$.

5) For $g \in G$, $g^{-1}g = 1_{\alpha(g)}$, $gg^{-1} = 1_{\beta(g)}$.

6) For $(g,h),(h,k) \in G \,_\alpha\times_\beta G$, $g(hk) = (gh)k$.

Above, the standard notation $\iota(g) = g^{-1}$, $\mu(g,h) = gh$ is used. The structural maps α, β, 1, ι, μ are called source, target, unit, inversion and partial multiplication, respectively. For $m \in M$, one sets $G_m = \alpha^{-1}(m)$, $G^m = \beta^{-1}(m)$ and, for $m, n \in M$, $G_m^n = G_m \cap G^n$.

A groupoid G over M is a *Lie groupoid* if G and M are smooth manifolds and the maps α, β, 1, ι, μ are smooth with α, β surjective submersion, 1 and injective immersion and ι a diffeomorphism. In what follows, we shall consider exclusively Lie groupoids.

The prototype Lie groupoid over M is the pair groupoid $G = M \times M$, whose structure maps are defined by $\alpha(m,n) = n$, $\beta(m,n) = m$, $1_m = (m,m)$, $(m,n)^{-1} = (n,m)$ and $(m,n)(n,p) = (m,p)$. Lie groupoids generalize Lie groups: a Lie group can be viewed as a Lie groupoid over the singleton manifold $M = \mathrm{pt}$.

A Lie groupoid G is called *regular* if, for each $m \in M$, the target map β restricts to a map $\beta : \alpha^{-1}(m) \to M$ of locally constant rank. A Lie groupoid G is called *transitive* if the map $(\alpha, \beta) : G \to M \times M$ is a surjective submersion. Every transitive Lie groupoid is regular.

A *(base preserving) morphism* of two Lie groupoids G, G' over M is a smooth map $F : G \to G'$ such that

1) $\alpha' \circ F = \alpha$, $\beta' \circ F = \beta$

3) For $(g,h) \in G \,_\alpha\times_\beta G$, $F(gh) = F(g)F(h)$.

If H, G are two Lie groupoids over M and H is an immersed submanifold of G, then H is a *Lie subgroupoid* of G if the natural injection $I : H \to G$ is a Lie groupoid morphism.

Let G be a regular Lie groupoid over M. Then, for $m \in M$, G_m^m is a Lie group, the *isotropy group* of m. The *isotropy groupoid* of G, I_G, is defined as the union of all isotropy groups of G:

$$I_G = \bigcup_{m \in M} G_m^m. \tag{6.1}$$

With the structural maps and the differential structure inherited from G, I_G is a Lie groupoid and a Lie subgroupoid of G. I_G is also a bundle of Lie groups.

Just as to any Lie group there is canonically associated a Lie algebra, to any Lie groupoid G over M there is canonically associated a Lie algebroid AG over M. Explicitly, one has

$$AG = \bigcup_{m \in M} T_{1_m} G_m \tag{6.2}$$

with the vector bundle structure induced by that of TG. The Lie algebroid structure of AG is defined as follows. A vector field $X \in \Gamma(TG)$ is said right invariant if: 1) for $g \in G$, $X(g) \in T_g G_{\alpha(g)}$; 2) for $(g,h) \in G \,_\alpha\times_\beta G$, $X(gh) = T_g R_h X(g)$, where, for $h \in G$, we define

$R_h(g) = gh$ with $g \in G_{\beta(h)}$. One can show that there is a one-to-one correspondence between sections of AG and right invariant vector fields of G defined by

$$\tilde{s}(g) = T_{1_{\beta(g)}} R_g \, s(1_{\beta(g)}), \qquad g \in G, \tag{6.3}$$

with $s \in \Gamma(AG)$. The Lie bracket of two right invariant vector fields of G is also right invariant. This allows to define the Lie bracket $[s, t]$ of two sections $s, t \in \Gamma(AG)$ through the relation

$$\widetilde{[s, t]} = [\tilde{s}, \tilde{t}]. \tag{6.4}$$

The anchor ρ is defined by

$$\rho(s)(m) = T_{1_m} \beta \, s(1_m), \tag{6.5}$$

for $m \in M$ and $s \in \Gamma(AG)$. It is straightforward to check that the basic relations (2.1)–(2.3) are satisfied.

For the pair groupoid $G = M \times M$, $AG = TM$. If G is a Lie group, then $AG = \mathfrak{g}$, the usual Lie algebra of G.

If G is a regular Lie groupoid, then AG is a regular Lie algebroid. Similarly, if G is a transitive Lie groupoid, then AG is a transitive Lie algebroid (cf. section 2).

Let G, G' be two Lie groupoids over M and let $F : G \to G'$ be a groupoid morphism. Then, setting

$$F_*(s)_m = T_{1_m} F \, s_m, \tag{6.6}$$

with $s_m \in T_{1_m} G_m$, defines a Lie algebroid morphism (cf. eq. (2.12), (2.13)).

If H is a Lie subgroupoid of G, then AH is a Lie subalgebroid of AG (cf. section 2). In particular, if G is a regular Lie groupoid, AI_G is a Lie subalgebroid of AG. In fact, one has $AI_G = \ker \rho$, as follows easily from (6.5).

Let G be a Lie groupoid over M and let $J : X \to M$ be a fibered manifold. A *left action* of G on X along J is a smooth map $\lambda : G_\alpha \times_J X \to X$, where $G_\alpha \times_J X = \{(g, x) \in G \times X | \alpha(g) = J(x)\}$ with the following properties.

1) For $(g, x) \in G_\alpha \times_J X$, $J(gx) = \beta(g)$.

2) For $x \in X$, $1_{J(x)} x = x$.

3) For $(g, h) \in G_\alpha \times_\beta G$, $(h, x) \in G_\alpha \times_J X$, $g(hx) = (gh)x$.

Above, the standard notation $\lambda(g, x) = gx$ is used.

To a left action of G on X along J, there is canonically associated an infinitesimal action of AG on X along J (cf. section 2). The associated map $u : \Gamma(AG) \to \Gamma(TX)$ is defined by

$$u(s)(x) = T_{1_{J(x)}} A_x \, s(1_{J(x)}), \qquad x \in G, \tag{6.7}$$

for $s \in \Gamma(AG)$, where, for $x \in X$, we define $A_x(g) = gx$ with $g \in G_{J(x)}$. It is a simple matter to check that the basic properties (2.14)–(2.16) of an infinitesimal action hold.

To a left action of G on X along J, there is canonically associated a Lie groupoid structure over X on the pull back $G_\alpha \times_J X$. The structural maps are defined by $s((g, x)) = x$,

$t((g, x)) = gx$ with $(g, x) \in G_\alpha \times_J X$, $1_x = (1_{J(x)}, x)$ with $x \in X$, $(g, x)^{-1} = (g^{-1}, gx)$ with $(g, x) \in G_\alpha \times_J X$ and and $(g, x)(h, y) = (gh, y)$ with (g, x), $(h, y) \in G_\alpha \times_J X$ such that $x = hy$. The resulting Lie groupoid is called the *action Lie groupoid* corresponding to the left action and is usually denoted by $G \ltimes J$.

It is an important result that $A(G \ltimes J) \simeq AG \ltimes J$: the Lie algebroid of the action Lie groupoid $G \ltimes J$ is isomorphic to the action Lie algebroid $AG \ltimes J$ (cf. section 2, eqs. (2.20), (2.21)).

Next, we discuss a generalization of Hamiltonian symmetry reduction for Lie groupoid actions on fibered Poisson manifolds. We follow closely the treatment of Bos in [44].[2]

Let G be a regular Lie groupoid over M acting on a fibered Poisson manifold $J : X \to M, P$ (cf. eqs. (2.22), (2.25)). P is said *invariant*, if it is invariant under the associated infinitesimal action of AG (cf. eq. (2.27)). For P invariant, the action is said Hamiltonian, if there exists an equivariant moment map μ for the AG action (cf. eqs. (2.29)–(2.31)). Henceforth, we assume that P is invariant and that the action is Hamiltonian with moment map μ.

As I_G is a Lie subgroupoid of G and $AI_G = \ker \rho$, one can view the moment map as a map $\mu : X \to (AI_G)^*$ such that $\pi_{AI_G} \circ \mu = J$, where $\pi_{AI_G} : (AI_G)^* \to M$ is the bundle projection. Let 0 denote the zero section of $(AI_G)^*$. Suppose that $0(M) \subset \operatorname{im} \mu$. Suppose further that, for each $m \in M$, the Lie group G_m^m is connected and that it acts freely and properly on $\mu^{-1}(0(m))$. Then, for each $m \in M$, the quotient manifold

$$X_m := G_m^m \backslash \mu^{-1}(0(m)) \tag{6.8}$$

is a smooth manifold. Now, note that $\mu^{-1}(0(m)) \subset J^{-1}(m)$. Then, since P is fibered, $P\big|_{J^{-1}(m)}$ is a Poisson structure on $J^{-1}(m)$ (cf. eq. (2.25)). Likewise, since the u is projectable, $u(z)\big|_{J^{-1}(m)}$ is a vector field on $J^{-1}(m)$, for all $z \in \Gamma(\ker \rho)$ (cf. eq. (2.16)). By the classic result of Marsden and Ratiu [34], X_m inherits a Poisson structure (Marsden-Weinstein quotient).

Suppose that μ and 0 are transversal, i.e. that, for any $m \in M$ and any $x \in X$ such that $\mu(x) = 0(m)$, $T_m 0(T_m M)$ and $T_x \mu(T_x X)$ are transversal in $T_{0(m)}(AI_G)^*$. Then, $\mu^{-1}(0(M))$ is a manifold. The map

$$\bigcup_{m \in M} X_m = I_G \backslash \mu^{-1}(0(M)) \to M \tag{6.9}$$

is a smooth family of Poisson manifolds.

Thus, under the assumption listed above, the Hamiltonian action of a regular Lie groupoid G over M on a Poisson manifold X fibered over M along the fibration leaving the Poisson structure invariant induces a family version of the Hamiltonian symmetry reduction. This reduction is encoded in the target space geometry of the Lie algebroid Poisson sigma model described in section 3, when the background Lie algebroid L is the Lie algebroid AG of the Lie groupoid G.

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[2]Actually, Bos considers only the symplectic case. Further, he uses the more precise term strongly internally Hamiltonian in place of Hamiltonian.

7. Examples

In this section, we shall illustrate a class of examples of the target space geometry of the Lie algebroid Poisson sigma model. The geometrical data are listed at the beginning of section 3.

Suppose that X is a vector bundle over M. There exists a vector bundle $\mathfrak{gl}(X)$ over M that fits in a short exact sequence of base preserving vector bundle morphisms of the form

$$0 \longrightarrow \operatorname{End} X \xrightarrow{\iota} \mathfrak{gl}(X) \xrightarrow{\varpi} TM \longrightarrow 0. \tag{7.1}$$

$\mathfrak{gl}(X)$ is isomorphic to the direct sum bundle $TM \oplus \operatorname{End} X$. The isomorphism is not canonical depending on the choice of a splitting, a vector bundle morphism $\sigma : TM \to \mathfrak{gl}(X)$ such that $\varpi \circ \sigma = \operatorname{id}_{TM}$. The splittings are in one to one correspondence with the connections of X. See [40] for background.

The details of the local description of the vector bundle $\mathfrak{gl}(X)$ are provided in app. B. We denote by m^r and $(\mu^r, \alpha^A{}_B)$ the local base and fiber coordinates of $\mathfrak{gl}(X)$. The morphisms ι and ϖ are then given locally by $\iota(m, \alpha) = (m^r, 0, \alpha^A{}_B)$ and $\varpi(m, \mu, \alpha) = (m^r, \mu^r)$, respectively.

$\mathfrak{gl}(X)$ has a natural structure of Lie algebroid. Its anchor is the morphism ϖ appearing in the sequence (7.1). Its Lie bracket is defined as follows. Let $s, t \in \Gamma(\mathfrak{gl}(X))$ be locally given as $s(m) \simeq (v^r(m), s^A{}_B(m))$, $t(m) \simeq (w^r(m), t^A{}_B(m))$, respectively. Then,

$$[s,t](m) \simeq ((v^s \partial_s w^r - w^s \partial_s v^r)(m), (v^s \partial_s t^A{}_B - w^s \partial_s s^A{}_B - s^A{}_C t^C{}_B + t^A{}_C s^C{}_B)(m)). \tag{7.2}$$

The Lie algebroid $\mathfrak{gl}(X)$ is transitive and thus regular.

To any section $s \in \Gamma(\mathfrak{gl}(X))$, there is associated a *linear vector field* $u(s)$ on X [40], that is a vector field depending linearly on the fiber coordinates of X, as follows. Let (m^r, e^A) and (μ^r, ϵ^A) be local base and fiber coordinates of TX, m^r and e^A being base and fiber coordinates of X (cf. app. B). If $s(m) \simeq (v^r(m), s^A{}_B(m))$ locally, then

$$u(s)(m, e) \simeq (v^r(m), s^A{}_B(m)e^B). \tag{7.3}$$

The linear vector fields form a Lie subalgebra $\Gamma(TX)_{\mathrm{lin}}$ of $\Gamma(TX)$ and the map $s \to u(s)$ defines a Lie algebra isomorphism $\Gamma(\mathfrak{gl}(X)) \simeq \Gamma(TX)_{\mathrm{lin}}$. Using this identification, one can view $\mathfrak{gl}(X)$ as a Lie algebroid whose sections are the linear vector fields of X.

To any section $s \in \Gamma(\mathfrak{gl}(X))$, there is associated a *derivative endomorphism* $\mathcal{D}_s$ of X. A derivative endomorphism is an $\mathbb{R}$-linear map $\mathcal{D} : \Gamma(X) \to \Gamma(X)$ such that there is a vector field $u_{\mathcal{D}} \in \Gamma(TM)$ such that

$$\mathcal{D}(f\sigma) = f\mathcal{D}\sigma + (l_{u_{\mathcal{D}}} f)\sigma \tag{7.4}$$

for $f \in C^\infty(M)$ and $\sigma \in \Gamma(X)$ [40]. If $s(m) \simeq (v^r(m), s^A{}_B(m))$ locally, then

$$\mathcal{D}_s \sigma^A(m) = (v^r \partial_r \sigma^A - s^A{}_B \sigma^B)(m). \tag{7.5}$$

Note that $u_{\mathcal{D}_s} = v$. The derivative endomorphism form a Lie algebra $\mathfrak{D}_X$ if the Lie bracket is defined as the usual operator commutator. The map $s \to \mathcal{D}_s$ defines a Lie algebra isomorphism $\Gamma(\mathfrak{gl}(X)) \simeq \mathfrak{D}_X$. Using this identification, one can view $\mathfrak{gl}(X)$ as a Lie algebroid whose sections are the derivative endomorphisms of X. Upon doing so, a representation of a Lie algebroid L over M on X can be regarded as a Lie algebroid morphism $D : L \to \mathfrak{gl}(X)$ (cf. section 4, eq. (4.3)).

As X is a vector bundle over M, $J : X \to M$ is a fibered manifold, J being the bundle projection. In local coordinates, $J^r(m, e) = m^r$.

If L is a Lie subalgebroid of $\mathfrak{gl}(X)$, then the map $s \in \Gamma(L) \to u(s) \in \Gamma(TX)$ defines an infinitesimal action of L on X along J (cf. section 2). Indeed, (2.14)–(2.16) are satisfied, as is easy to check from (7.2), (7.3).

If X is fibered Poisson manifold and $P \in \Gamma(\wedge^2 TX)$ is its Poisson structure, then (2.26) holds. In local coordinates, this yields the equations

$$P^{rs} = 0, \qquad P^{rA} = 0. \tag{7.6}$$

Only the components P^{AB} may be non zero. Thus, actually $P \in \Gamma(\wedge^2 \operatorname{Vert} TX)$, where $\operatorname{Vert} TX = J^* X$ is the vertical subbundle of TX. The Poisson condition (2.24) obeyed by P reduces then into

$$P^{AD} \partial_D P^{BC} + P^{BD} \partial_D P^{CA} + P^{CD} \partial_D P^{AB} = 0. \tag{7.7}$$

If $\partial_C P^{AB} = 0$, (7.7) is automatically satisfied. In that case, one can view $P \in \Gamma(\wedge^2 X)$.

Let $s \in \Gamma(\mathfrak{gl}(X))$ and let P be invariant under the linear vector field $u(s)$. Then, (2.28) holds. Explicitly, if $s \simeq (v^r, s^A{}_B)$, one has

$$v^r \partial_r P^{AB} - s^A{}_C P^{CB} - s^B{}_C P^{AC} + s^C{}_D e^D \partial_C P^{AB} = 0. \tag{7.8}$$

If $\varpi(s) = 0$, then $u(s)$ is Hamiltonian if

$$v^r = 0 \tag{7.9a}$$

$$s^A{}_B e^B = -P^{AB} \partial_B \mu(s), \tag{7.9b}$$

for some function $\mu(s) \in C^\infty(X)$.

In general, the sections $s \in \Gamma(\mathfrak{gl}(X))$ such that (7.8) holds are not sections of some regular Lie subalgebroid L of $\mathfrak{gl}(X)$. If such an L can be found, however, then the infinitesimal action of L on X leaves P invariant (cf. section 2, eq. (2.27)). Even when L does exist, in general $u(s)$ is not Hamiltonian for $s \in \Gamma(\ker \rho)$, where $\rho = \varpi|_L$ is the anchor of L. We do not know any general condition ensuring the existence of L and the Hamiltonianity of its action on X. Below, we present a possible scenario where this can happen.

Suppose that $P \in \Gamma(\wedge^2 TX)$ satisfies (7.6) and the linearity condition

$$P^{AB}(m, e) = \pi^{AB}{}_C(m) e^C, \tag{7.10}$$

where $\pi \in \Gamma(\wedge^2 X \otimes X^*)$. The Poisson condition (7.7) then becomes a purely algebraic relation

$$\pi^{AD}{}_E \pi^{BC}{}_D + \pi^{BD}{}_E \pi^{CA}{}_D + \pi^{CD}{}_E \pi^{AB}{}_D = 0. \tag{7.11}$$

Thus, the dual bundle X^* of X is a bundle of Lie algebras.

If $s \in \Gamma(\mathfrak{gl}(X))$ with $s \simeq (v^r, s^A{}_B)$ locally, (7.8) is satisfied if and only if

$$v^r \partial_r \pi^{AB}{}_C - s^A{}_D \pi^{DB}{}_C - s^B{}_D \pi^{AD}{}_C + s^D{}_C \pi^{AB}{}_D = 0. \tag{7.12}$$

This condition is purely algebraic in s. Thus, it defines a subspace in each fiber of $\mathfrak{gl}(X)$. With some regularity assumption on π made, this distribution of subspaces is a subbundle L of $\mathfrak{gl}(X)$. Since $l_{u(s)}P = l_{u(t)}P = 0$ implies $l_{u([s,t])}P = 0$, L is in fact a Lie subalgebroid of $\mathfrak{gl}(X)$. L then acts infinitesimally on X along J leaving P invariant, by construction.

If $s \in \Gamma(\ker \rho)$, then $v^r = 0$ and (7.12) becomes

$$s^A{}_D \pi^{DB}{}_C + s^B{}_D \pi^{AD}{}_C - s^D{}_C \pi^{AB}{}_D = 0. \tag{7.13}$$

Then, for every $m \in M$, $s(m)$ is a 1-cocycle of the Lie algebra cohomology of $X^*{}_m$ with values in $\operatorname{ad} X^*{}_m$. If

$$s^A{}_B = -\pi^{AC}{}_B t_C \tag{7.14}$$

for some $t \in \Gamma(X^*)$, this 1-cocycle is a 1-coboundary. In such case, $u(s)$ is Hamiltonian: (7.9b) is fulfilled with

$$\mu(s)(m, e) = t_A(m) e^A. \tag{7.15}$$

In order this to be the case, it suffices to require that the 1st Lie algebra cohomology of $X^*{}_m$ vanishes for all $m \in M$.

The geometrical setup described above is automatically integrable. Let $GL(X)$ be the set of all linear isomorphisms $T : X_m \to X_n$ with $m, n \in M$. Then, $GL(X)$ has a natural structure of Lie groupoid over M: $\alpha(T) = m$, $\beta(T) = m$, for $T : X_m \to X_n$; $1_m = \operatorname{id}_{X_m}$; the inversion and partial multiplication are the corresponding operations for linear isomorphisms. It can be shown that $AGL(X) \simeq \mathfrak{gl}(X)$ [40, 42]. Thus, $\mathfrak{gl}(X)$ is automatically integrable and so is every Lie subalgebroid L of $\mathfrak{gl}(X)$.

8. Concluding remarks

In this final section, we briefly review and comment on the problems which are still open.

When $M = \operatorname{pt}$ and $J : X \to M$ is the constant map, the Lie algebroid L is an ordinary Lie algebra acting infinitesimally on X leaving the Poisson structure P invariant. As noticed in section 3, in this case the Lie algebroid Poisson sigma model reduces into the Poisson-Weil model of refs. [25, 26] for trivial twisting bundle. The twisting bundle is a principal bundle on the world sheet Σ with structure group G integrating the Lie algebra L. It is the gauge bundle of the Poisson-Weil sigma model as a 2-dimensional gauge theory. The natural question arises whether it is possible to generalize the construction described in the present work in such a way to recover, in the Lie algebra case, the Poisson-Weil model with arbitrary twisting bundle. Presumably, this requires the following.

1. The symmetry of the target space geometry is encoded in a Lie groupoid G over M integrating L.

2. The twisting structure is a principal groupoid bundle P with base Σ and structure groupoid G.

Recall that a *principal groupoid bundle* P over a manifold Σ is a smooth fiber bundle $\pi : P \to \Sigma$ endowed with a smooth right action μ of G along $\kappa : P \to M$ preserving the fibers of P and such that the map

$$(\mathrm{pr}_1, \mu) : P_\kappa \times_\beta G \to P_\pi \times_\pi P : (p, g) \mapsto (p, pg) \tag{8.1}$$

is a diffeomorphism. Groupoid right actions are defined in a way totally analogous to left actions (cf. section 6). Diagrammatically, the bundle can be represented as

$$
\begin{array}{ccc}
P & & G \\
{\scriptstyle \pi}\downarrow & {\scriptstyle \kappa}\searrow \ {\scriptstyle \alpha}\downarrow\downarrow{\scriptstyle \beta} & \\
\Sigma & & M
\end{array}
\quad .
$$

It can be shown that when $M = \mathrm{pt}$ and G is a Lie group, one recovers the customary notion of principal (group) bundle. See [24] and references therein for background.

Unfortunately, at present, it is not clear to us how to implement this more general form of twisting in a Lagrangian field theoretic framework essentially because we do not know how to build objects globally defined on Σ which can be integrated out of the above geometrical data. This is an open issue calling for further investigation.

As is well known, the BV master action of a field theory is not directly usable for quantization: gauge fixing is required. Fixing the gauge consists in restricting to a suitable Lagrangian submanifold if field space. It is notoriously a very difficult problem. Normally, it can be done only in certain cases, when the background geometry has extra structures, and there are no general methods for its implementation.

For the Poisson-Weil model, gauge fixing has been worked out by us in [26], taking inspiration from the classical work of AKSZ [37], and has led to interesting topological field theories such as the 2-dimensional Donaldson-Witten topological gauge theory [28, 29] and the gauged A topological sigma model [30 – 32]. At the moment, we know no sensible gauge fixing prescriptions of the Lie algebroid Poisson sigma model yielding interesting topological field theories. As far as we know, there may not be any.

Generalized complex geometry [45, 46] has been the object of much interest in recent years for its role in superstring flux compactifications [47]. In [48 – 50], following the AKSZ philosophy of [38, 39] and extending the Poisson sigma model, we introduced a BV field theoretic realization of generalized complex geometry, the Hitchin sigma model, and in [25], we gauged it by coupling it to the Weil model. It would be interesting to generalize the construction of the present paper to the Hitchin model. The target space geometry of the "Lie algebroid Hitchin model" is expected to be extremely rich and interesting.

A. Analysis of covariance I

In this appendix, we shall present an analysis of covariance for the cotangent bundle $T[1]^* \mathfrak{X}_{L,J}$ of the manifold $\mathfrak{X}_{L,J}$ defined in (3.1).

Recall that L is a regular Lie algebroid over M acting infinitesimally on a fibered manifold $J : X \to M$. Since $\ker \rho$ is a subbundle of L, we conveniently use trivializations of L adapted to $\ker \rho$. Thus, the fiber coordinates $\{v^i\}$ of any vector $v \in L$ get subdivided as $\{v^\alpha\} \cup \{v^\kappa\}$ and $v \in \ker \rho$ if and only if $v^\kappa = 0$ for all κ. We denote by $(T^{i'}{}_j)$ the transition matrix function of a generic change of adapted trivialization of L. The upper left block of $(T^{i'}{}_j)$, $(T^{\alpha'}{}_\beta)$, is the transition matrix function of the associated change of trivialization of $\ker \rho$.

We denote by ξ^a, $(\beta_i, \mathrm{B}_\alpha)$ respectively the base and fiber coordinates of the vector bundle $\mathfrak{X}_{L,J} = (J^*L)^*[0] \oplus (J^* \ker \rho)^*[-1]$ with respect to some trivialization. Then, the cotangent bundle $T^*[1]\mathfrak{X}_{L,J}$ has base coordinates $(\xi^a, \beta_i, \mathrm{B}_\alpha)$ and fiber coordinates $(\eta_a, \gamma^i, \Gamma^\alpha)$.

A straightforward differential geometric analysis shows that under a change of trivialization, one has

$$\xi^{a'} = F^{a'}(\xi), \tag{A.1a}$$
$$\beta_{i'} = T^{-1j}{}_{i'}(J'(F(\xi)))\beta_j, \tag{A.1b}$$
$$\mathrm{B}_{\alpha'} = T^{-1\beta}{}_{\alpha'}(J'(F(\xi)))\mathrm{B}_\beta. \tag{A.1c}$$
$$\eta_{a'} = \partial_{a'}F^{-1b}(F(\xi))\big[\eta_b + T^{-1i}{}_{k'}(J'(F(\xi)))\partial_b J^r(\xi)\partial_r T^{k'}{}_j(J(\xi))\beta_i\gamma^j \tag{A.1d}$$
$$+ T^{-1\alpha}{}_{\gamma'}(J'(F(\xi)))\partial_b J^r(\xi)\partial_r T^{\gamma'}{}_\beta(J(\xi))\mathrm{B}_\alpha\Gamma^\beta\big],$$
$$\gamma^{i'} = T^{i'}{}_j(J(\xi))\gamma^j, \tag{A.1e}$$
$$\Gamma^{\alpha'} = T^{\alpha'}{}_\beta(J(\xi))\Gamma^\beta, \tag{A.1f}$$

Under a change of trivialization, the anchor and structure functions transform as follows

$$u_{i'}{}^{a'}(\xi') = \partial_b F^{a'}(\xi)T^{-1j}{}_{i'}(J'(F(\xi)))u_j{}^b(\xi), \tag{A.2}$$
$$f^{k'}{}_{i'j'}(J'(\xi')) = T^{k'}{}_n(J(\xi))T^{-1l}{}_{i'}(J'(F(\xi)))T^{-1m}{}_{j'}(J'(F(\xi))) \tag{A.3}$$
$$\times \big[f^n{}_{lm}(J(\xi)) - T^{-1n}{}_{h'}(J'(F(\xi)))u_l{}^a(\xi)\partial_a J^r(\xi)\partial_r T^{h'}{}_m(J(\xi))$$
$$+ T^{-1n}{}_{h'}(J'(F(\xi)))u_m{}^a(\xi)\partial_a J^r(\xi)\partial_r T^{h'}{}_l(J(\xi))\big].$$

Exploiting the above relations, it is straightforward though lengthy to verify the target space global definedness of the sigma model action (3.4).

B. Analysis of covariance II

In this appendix, we shall present an analysis of covariance for the vector bundle $\mathfrak{gl}(X)$ studied in section 7.

Let X be a vector bundle over M. Let m^r, e^A be the base and fiber coordinates associated with a given trivialization of X, respectively. Under a change of trivialization, they transform as

$$m^{r'} = \Phi^{r'}(m), \tag{B.1a}$$
$$e^{A'} = \Theta^{A'}{}_B(m)e^B, \tag{B.1b}$$

where $(\Theta^{A'}{}_B)$ is the transition matrix function of the trivialization change.

Consider next the vector bundle TX. To each trivialization of X, there corresponds one of TX, with associated base and fiber coordinates (m^r, e^A), (μ^r, ϵ^A), respectively. Under a change of trivialization, one has (B.1) and

$$\mu^{r'} = \partial_s \Phi^{r'}(m)\mu^s, \tag{B.2a}$$

$$\epsilon^{A'} = \partial_r \Theta^{A'}{}_B(m)\mu^r e^B + \Theta^{A'}{}_B(m)\epsilon^B. \tag{B.2b}$$

TX is actually a "double vector bundle" [40]: it is not only a vector bundle over X but also one over TM. This becomes apparent upon considering (m^r, μ^r) as base coordinates and (e^A, ϵ^A) as fiber coordinates.

The vector bundle $\mathfrak{gl}(X)$ can be described locally by specifying an atlas of local coordinates together with the coordinate change transformation relations. To each trivialization of X, there corresponds one of $\mathfrak{gl}(X)$ with base coordinates m^r and fiber coordinates μ^r, $\alpha^A{}_B$, transforming according to (B.1a), (B.2a) and

$$\alpha^{A'}{}_{B'} = \partial_r \Theta^{A'}{}_C(m)\Theta^{-1C}{}_{B'}(\Phi(m))\mu^r + \Theta^{A'}{}_C(m)\alpha^C{}_D\Theta^{-1D}{}_{B'}(\Phi(m)). \tag{B.3}$$

A connection of X is given in a trivialization by local 1-forms $A_r{}^A{}_B(m)dm^r$ transforming under the change of trivialization (B.1) as

$$A_{r'}{}^{A'}{}_{B'}(m') = \partial_{r'}\Phi^{-1s}(\Phi(m))\big[-\partial_s\Theta^{A'}{}_C(m)\Theta^{-1C}{}_{B'}(\Phi(m)) \tag{B.4}$$
$$+ \Theta^{A'}{}_C(m)A_s{}^C{}_D(m)\Theta^{-1D}{}_{B'}(\Phi(m))\big].$$

Upon picking a connection, there is defined a vector bundle isomorphism $\mathfrak{gl}(X) \simeq TM \oplus \operatorname{End} X$ locally defined by $(\mu^r, \alpha^A{}_B) \to (\mu^r, \bar{\alpha}^A{}_B)$, where

$$\bar{\alpha}^A{}_B = A_r{}^A{}_B(m)\mu^r + \alpha^A{}_B. \tag{B.5}$$

C. Action Lie algebroid Poisson cohomology

In this appendix, we present a supergeometric description of the action Lie algebroid Poisson double complex $(A_J{}^{*,*}(L), d_{J,L}, d_P)$ introduced in section 4. The following construction is based on the graded vector bundle $J^*L[1] \oplus T^*[1]X$ with base X. $A_J{}^{*,*}(L)$ and $d_{J,L}$, d_P are then realized as a subspace of functions on $J^*L[1] \oplus T^*[1]X$ and as degree 1 vector fields on $J^*L[1] \oplus T^*[1]X$, respectively.

Denote by ξ^a and (γ^i, η_a) the base and odd fiber coordinates of $J^*L[1] \oplus T^*[1]X$, respectively. Then, a generic function $\Phi \in C^\infty(J^*L[1] \oplus T^*[1]X)$ has the form

$$\Phi = \sum_{p,q \geq 0} \frac{1}{p!q!}\phi^{(p,q)}{}_{i_1\ldots i_p}{}^{a_1\ldots a_q}(\xi)\gamma^{i_1}\ldots\gamma^{i_p}\eta_{a_1}\ldots\eta_{a_q} \tag{C.1}$$

with $\phi^{(p,q)} \in \Gamma(\wedge^p(J^*L)^* \otimes \wedge^q TX)$. Now, define the degree -1 vector fields on $J^*L[1] \oplus T^*[1]X$

$$K^r = \partial_a J^r(\xi)\partial_\eta{}^a, \tag{C.2}$$

where $\partial_\eta{}^a = \partial/\partial\eta_a$. So, recalling that $A_J{}^{p,r}(L) = \Gamma(\wedge^p(J^*L)^* \otimes \wedge^r T^J X)$, it appears that $A_J{}^{*,*}(L)$ is identified with the intersection of the kernels of the K^r.

By inspection, one can check that the differentials $d_{J,L}$, d_P of $A_{J^*,*}(L)$ are then identified with the degree 1 vector fields on $J^*L[1] \oplus T^*[1]X$

$$d_{J,L} = u_i{}^a(\xi)\gamma^i\partial_a - \partial_b u_i{}^a(\xi)\gamma^i\eta_a\partial_\eta{}^b - \frac{1}{2}f^k{}_{ij}(J(\xi))\gamma^i\gamma^j\partial_{\gamma k}, \qquad (C.3)$$

$$d_P = -P^{ab}(\xi)\eta_a\partial_b + \frac{1}{2}\partial_c P^{ab}(\xi)\eta_a\eta_b\partial_\eta{}^c, \qquad (C.4)$$

where $\partial_{\gamma i} = \partial/\partial\gamma^i$. These satisfy the graded commutation relations

$$[d_{J,L}, K^r] = \partial_s\rho_i{}^r(J(\xi))\gamma^i K^s, \qquad (C.5a)$$

$$[d_P, K^r] = 0, \qquad (C.5b)$$

$$[d_{J,L}, d_{J,L}] = -\partial_r f^k{}_{ij}(J(\xi))u_k{}^a(\xi)\gamma^i\gamma^j\eta_a K^r, \qquad (C.5c)$$

$$[d_{J,L}, d_P] = 0, \qquad (C.5d)$$

$$[d_P, d_P] = 0, \qquad (C.5e)$$

as can be checked using (2.8), (2.18), (2.19), (2.24), (2.26), (2.28). Therefore, $d_{J,L}$, d_P preserve the subspace of functions $A_{J^*,*}(L)$ and are nilpotent and anticommute on it, as they should.

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